

# TREE-SHIFTS: THE ENTROPY OF TREE-SHIFTS OF FINITE TYPE

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**ABSTRACT.** This paper studies the entropy of tree-shifts of finite type with and without boundary conditions. We demonstrate that computing the entropy of a tree-shift of finite type is equivalent to solving a system of nonlinear recurrence equations. Furthermore, the entropy of the binary Markov tree-shifts over two symbols is either 0 or  $\ln 2$ . Meanwhile, the realization of a class of reals including multinacci numbers is elaborated, which indicates that tree-shifts are capable of rich phenomena. By considering the influence of three different types of boundary conditions, say, the periodic, Dirichlet, and Neumann boundary conditions, the necessary and sufficient conditions for the coincidence of entropy with and without boundary conditions are addressed.

## 1. INTRODUCTION

**1.1. History and motivation.** Shift space is a powerful tool to describe the stationary solutions in physical systems and lattice dynamical systems. Namely, for translation invariant lattice dynamical systems, stationary solutions could be generated by posing some local rules in the lattice. One can study the topological behavior of such systems by studying the shift space instead. The same idea is also used in dynamical systems. Namely, a map which admits Markov partition is semi-conjugate to a shift space. The investigation of the topological properties of such a map is equivalent to exploring the corresponding shift.

In the classical symbolic dynamical systems, shifts of finite type are an important class for the purpose of illustration. A shift of finite type is realized as a set of infinite paths in a finite directed graph. In the past three decades, the study of the complexity for the given systems has received considerable attention. Entropy plays an important role to measure such

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a property. In information theory, entropy also measures the “information capacity” or “ability to transmit messages”. Readers are referred to [3, 13, 20, 21, 25] and the references therein for more details.

While the dynamical behavior of shifts of finite type is explicitly characterized, the properties of multidimensional symbolic dynamical systems are barely known. For the classical symbolic dynamics, Williams indicates that the conjugacy of one-sided shifts of finite type is decidable, and topological entropy can be treated as an indicator for classifying shifts of finite types (see [21]). Nevertheless, the conjugacy of multidimensional shifts of finite type is undecidable (see [12, 18, 22]). Furthermore, there is no algorithm for the computation of topological entropy of multidimensional shifts of finite type (see [7, 9, 16, 24] and the references therein).

In [4, 5], the authors introduce the notion of shifts defined on infinite trees, that are called tree-shifts. Infinite trees have a natural structure of one-sided symbolic dynamical systems equipped with multiple shift maps. The  $i$ th shift map applies to a tree which gives the subtree rooted at the  $i$ th children of the tree. Sets of finite patterns of tree-shifts of finite type are strictly testable tree languages. Such testable tree languages are also called  $k$ -testable tree languages. Probabilistic  $k$ -testable models are used for pattern classification and stochastic learning [28].

Tree-shifts are interesting for elucidation due to them constituting an intermediate class in between one-sided shifts and multidimensional shifts. Aubrun and Béal extend Williams’ result to tree-shifts; more precisely, they show that the conjugacy of irreducible tree-shifts of finite type is decidable [4]. Furthermore, Aubrun and Béal accomplish other celebrated results in tree-shifts, such as realizing tree-shifts of finite type and sofic tree-shifts via tree automata, developing an algorithm for determining whether a sofic tree-shift is a tree-shift of finite type, and the existence of irreducible sofic tree-shifts that are not factors of tree-shifts of finite type. Readers are referred to [4, 5] for more details.

This paper, which is a consequential investigation of [6], focuses on the entropy of tree-shifts of finite type and intends to reveal the connection between tree-shifts of finite type and nonlinear dynamical systems. We demonstrate that the characterization of entropy of tree-shifts of finite type relates to solving systems of nonlinear recurrence equations (Theorems 1.5 and 1.6). Meanwhile, the entropy of binary Markov tree-shifts over the alphabet of two symbols is elaborated completely (Theorem 2.10) while a

large set of reals can be realized via Markov tree-shifts over an alphabet of more than two symbols (Theorem 3.1).

Notably, the investigation of entropy of shifts of finite type is mainly relevant to the rules of allowed patterns and can be treated as a subclass of recurrence equations. The computation is rather difficult for multidimensional shifts. In other words, the effect of dimension of the underlying spaces is only seen in the higher dimensional cases. The difficulty also surrounds the elucidation of entropy of tree-shifts of finite type. More explicitly, computing entropy of tree-shifts is equivalent to solving systems of nonlinear recurrence equations, but it is known that the computation of entropy of shifts of finite type is investigating linear recurrence equations instead. Rather than the novelty of elaboration of systems of nonlinear recurrence equations, there are many discussion about nonlinear recurrence equations, readers are referred to [14] for more details. Remarkably, a  $d$ -dimensional tiling system is a tree-shifts of finite type with abelian nodes and each node has  $d$  children. The related discussion is in preparation.

Meanwhile, boundary conditions are commonly considered in lattice dynamical systems and three types are typically considered. Namely, *Dirichlet* (or *fixed*), *Neumann* (or *zero-flux*), and *periodic* boundary conditions. (See [8, 10, 11, 15, 19, 23, 26, 27] for examples.) Traditionally, a Dirichlet condition means that such a system is forced to have a fixed input and output by energy in the boundary; a Neumann condition means that there is no input in the boundary cell due to the fact that any input would cause energy and/or material flow from the outside making the system an “open system” in the sense of thermodynamics in physics; the periodic boundary condition is equivalent to fabricating a chip on a silicon torus as its substrate.

For a system with a boundary constraint, it is interesting to see the effect from various boundary conditions. Correspondingly, the following question arose:

$$(1) \quad h = h^N = h^P = h^D?$$

Herein,  $h$  relates to the entropy, and  $h^N, h^P$ , and  $h^D$  relates to the entropy constrained with Neumann, periodic, and Dirichlet boundary conditions, respectively. Such a problem was posed in [1] and the references therein. Intrinsically, it indicates that the complexity of a given system will not be influenced by the boundary if the equality holds. Otherwise it will be influenced and  $h - h^i$  measures the difference. In circuit theory, it also means that we could control the “ideal system” (i.e. without a boundary

condition) by manipulating the input in the boundary (see [11] for more details). Theorem 4.1 (resp. Theorem 4.2 and Theorem 4.3) provides a natural and intrinsic characterization of question (1) for  $h = h^N$  (resp.  $h = h^D$  and  $h = h^P$ ).

The upcoming subsection elucidates the correspondence between Markov tree-shifts and systems of nonlinear recurrence equations, followed by the analysis of the entropy of binary Markov tree-shifts with two symbols. Section 3 addresses the realization theorem of tree-shifts of finite type. The boundary influence for the entropy is elucidated in Section 4. Discussion and conclusions are given in Section 5.

**1.2. Preliminaries and notations.** This subsection recalls some basic definitions of symbolic dynamics on infinite trees. The nodes of infinite trees considered in this paper have a fixed number of children and are labeled in a finite alphabet. Hence the class of classical one-sided shift spaces is a special case in the present paper.

Let  $\Sigma = \{0, 1, \dots, d-1\}$  and let  $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$  be the set of words over  $\Sigma$ , where  $\Sigma^n = \{w_1 w_2 \cdots w_n : w_i \in \Sigma \text{ for } 1 \leq i \leq n\}$  is the set of words of length  $n$  for  $n \in \mathbb{N}$  and  $\Sigma^0 = \{\epsilon\}$  consists of the empty word  $\epsilon$ . An *infinite tree*  $t$  over a finite alphabet  $\mathcal{A}$  is a function from  $\Sigma^*$  to  $\mathcal{A}$ . A node of an infinite tree is a word of  $\Sigma^*$ . It is seen that the empty word relates to the root of the tree. Suppose  $x$  is a node of a tree.  $x$  has children  $xi$  with  $i \in \Sigma$ . A sequence of words  $(x_k)_{1 \leq k \leq n}$  is called a *path* if, for all  $k \leq n-1$ ,  $x_{k+1} = x_k i_k$  for some  $i_k \in \Sigma$ .

Let  $t$  be a tree and let  $x$  be a node, we refer  $t_x$  to  $t(x)$  for simplicity. A subset of words  $L \subset \Sigma^*$  is called *prefix-closed* if each prefix of  $L$  belongs to  $L$ . A function  $u$  defined on a finite prefix-closed subset  $L$  with codomain  $\mathcal{A}$  is called a *pattern* (or *block*), and  $L$  is called the *support* of the pattern. A subtree of a tree  $t$  rooted at a node  $x$  is the tree  $t'$  satisfying  $t'_y = t_{xy}$  for all  $y \in \Sigma^*$  such that  $xy$  is a node of  $t$ , where  $xy = x_1 \cdots x_m y_1 \cdots y_n$  means the concatenation of  $x = x_1 \cdots x_m$  and  $y_1 \cdots y_n$ .

Suppose  $n$  is a nonnegative integer. Let  $\Sigma_n = \bigcup_{k=0}^n \Sigma^k$  denote the set of words of length at most  $n$ . We say that a pattern  $u$  is a *block of height  $n$*  (or *an  $n$ -block*) if the support of  $u$  is  $\Sigma_{n-1}$ , denoted by  $\text{height}(u) = n$ . Furthermore,  $u$  is a pattern of a tree  $t$  if there exists  $x \in \Sigma^*$  such that  $u_y = t_{xy}$  for every node  $y$  of  $u$ . In this case, we say that  $u$  is a pattern of  $t$  rooted at the node  $x$ . A tree  $t$  is said to *avoid*  $u$  if  $u$  is not a pattern of  $t$ . If  $u$  is a pattern of  $t$ , then  $u$  is called an *allowed pattern* of  $t$ .

We denote by  $\mathcal{T}$  (or  $\mathcal{A}^{\Sigma^*}$ ) the set of all infinite trees on  $\mathcal{A}$ . For  $i \in \Sigma$ , the shift transformations  $\sigma_i$  from  $\mathcal{T}$  to itself are defined as follows. For every tree  $t \in \mathcal{T}$ ,  $\sigma_i(t)$  is the tree rooted at the  $i$ th child of  $t$ , that is,  $\sigma_i(t)_x = t_{ix}$  for all  $x \in \Sigma^*$ . For the simplification of the notation, we omit the parentheses and denote  $\sigma_i(t)$  by  $\sigma_i t$ . The set  $\mathcal{T}$  equipped with the shift transformations  $\sigma_i$  is called the *full tree-shift* of infinite trees over  $\mathcal{A}$ . Suppose  $w = w_1 \cdots w_n \in \Sigma^*$ . Define  $\sigma_w = \sigma_{w_n} \circ \sigma_{w_{n-1}} \circ \cdots \circ \sigma_{w_1}$ . It follows immediately that  $(\sigma_w t)_x = t_{wx}$  for all  $x \in \Sigma^*$ .

Given a collection of patterns  $\mathcal{F}$ , let  $X_{\mathcal{F}}$  denote the set of all trees avoiding any element of  $\mathcal{F}$ . A subset  $X \subseteq \mathcal{T}$  is called a *tree-shift* if  $X = X_{\mathcal{F}}$  for some  $\mathcal{F}$ . We say that  $\mathcal{F}$  is a *set of forbidden patterns* (or a *forbidden set*) of  $X$ . It can be seen that a tree-shift satisfies  $\sigma_w X \subseteq X$  for all  $w \in \Sigma^*$ . A tree-shift  $X = X_{\mathcal{F}}$  is called a *tree-shift of finite type* (TSFT) if the forbidden set  $\mathcal{F}$  is finite.

Denote the set of all blocks of height  $n$  of  $X$  by  $B_n(X)$ , and denote the set of all blocks of  $X$  by  $B(X)$ . Suppose  $u \in B_n(X)$  for some  $n \geq 2$ . Let  $\sigma_i u$  be the block of height  $n-1$  such that  $(\sigma_i u)_x = u_{ix}$  for  $x \in \Sigma_{n-2}$ . The block  $u$  is written as  $u = (u_\epsilon, \sigma_0 u, \sigma_1 u, \dots, \sigma_{d-1} u)$ .

Let  $\mathcal{B}$  be a collection of 2-blocks, and let  $X^{\mathcal{B}} = X_{\mathcal{F}}$ , where  $\mathcal{F} = B_2(\mathcal{T}) \setminus \mathcal{B}$ . It follows immediately that  $X^{\mathcal{B}}$  is a TSFT. We say that  $\mathcal{B}$  is a *basic set of allowed patterns* (or a *basic set*) of  $X^{\mathcal{B}}$  and  $X^{\mathcal{B}}$  is generated by  $\mathcal{B}$ . It can be seen that a tree-shift satisfies  $\sigma_w X^{\mathcal{B}} \subseteq X^{\mathcal{B}}$  for all  $w \in \Sigma^*$ . In particular, for the case where  $d = 2$ , a 2-block  $u \in \mathcal{B}$  is written as  $(i, j, k)$ .

We remark that a basic set of allowed patterns is, in general, a subset of  $B_n(\mathcal{T})$  for some  $n \in \mathbb{N}$ . For the clarification of the investigation, we consider  $\mathcal{B} \subseteq B_2(\mathcal{T})$  for the rest of this paper unless otherwise stated. A TSFT is called a *Markov tree-shift* in this case. One of the most frequently used quantum which describes the complexity of a dynamical system is *entropy*. For a tree-shift  $X^{\mathcal{B}}$ , the definition of entropy is given as follows.

**Definition 1.1.** Suppose a basic set  $\mathcal{B}$  is given.

- (1) The *entropy* of  $X^{\mathcal{B}}$ , denoted by  $h(X^{\mathcal{B}}) = h(\mathcal{B})$ , is defined as

$$(2) \quad h(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln^2 |B_n(X^{\mathcal{B}})|}{n},$$

whenever the limit exists, where  $|B_n(X^{\mathcal{B}})|$  means the cardinality of  $B_n(X^{\mathcal{B}})$ , and  $\ln^2 = \ln \circ \ln$ .

- (2) If  $|B_n(X^{\mathcal{B}})|$  behaves like  $\exp(\alpha \kappa^n)$ , such as  $c_1 \exp(\alpha \kappa^n) \leq |B_n(X^{\mathcal{B}})| \leq c_2 \exp(\alpha \kappa^n)$  for some  $c_1, c_2 > 0$  and  $\alpha, \kappa \geq 0$  are constants. From (2)

we have  $h(\mathcal{B}) = \ln \kappa$ , and called  $\alpha$  the *hidden entropy* (or *sub-entropy*) of  $X^{\mathcal{B}}$ .

It is seen that the topological entropy of a full tree-shift  $\mathcal{T} = \mathcal{A}^{\Sigma^*}$  is  $h(\mathcal{T}) = \ln d$ , where  $d = |\Sigma| \in \mathbb{N}$  with  $d \geq 2$ . Indeed, the cardinality of  $n$ -blocks in  $\mathcal{T}$  is  $|B_n(\mathcal{T})| = k^{\frac{d^n-1}{d-1}}$ , where  $k = |\mathcal{A}|$ . It follows that

$$h(\mathcal{T}) = \lim_{n \rightarrow \infty} \frac{\ln^2 |B_n(\mathcal{T})|}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 k + \ln(d^n - 1) - \ln(d - 1)}{n} = \ln d.$$

The following lemma addresses the sufficient condition for the existence of the limit (2). The proof is straightforward, thus it is omitted.

**Lemma 1.2.** *The limit (2) exists if  $\lim_{n \rightarrow \infty} \frac{\ln |B_n(X^{\mathcal{B}})|}{d^n} > 0$ .*

Aside from the existence of the limit (2), it is natural to ask the following question.

**Problem 1.** How can the hidden entropy be computed?

This work focuses on the investigation of (2). The study of Problem 1 is in preparation. It is seen below that, for those “good enough” basic sets  $\mathcal{B}$ , the entropy  $h(\mathcal{B})$  can be computed explicitly.

Suppose  $X^{\mathcal{B}}$  is a one-sided shift space, i.e., consider the case where  $d = 1$ . The entropy of  $X^{\mathcal{B}}$  is defined as

$$h(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln |B_n(X^{\mathcal{B}})|}{n}.$$

Notably, in this case,  $B_n(X^{\mathcal{B}})$  is a subset of  $\mathcal{A}^n$ . This makes the growth rate of  $|B_n(X^{\mathcal{B}})|$  with respect to  $n$  up to exponential, and the existence of the limit comes from the subadditivity of  $\{\ln |B_n(X^{\mathcal{B}})|\}_{n \geq 1}$  (see [21]). For the case where  $X^{\mathcal{B}}$  is a strict tree-shift, i.e.,  $d \geq 2$ , the growth rate of  $|B_n(X^{\mathcal{B}})|$  with respect to  $n$  becomes doubly exponential since the size of the support of an  $n$ -block increases doubly exponentially.

Another difference between the topological entropy of one-sided shift spaces and tree-shifts is the introduction of hidden entropy. It is remarkable that, for a strict tree-shift rather than a one-sided shift space, the hidden entropy means the “average number of symbols” used while the topological entropy infers the “average spatial dimension” of the underlying lattice. An intuitive observation can be seen from the full tree-shift. Suppose  $\mathcal{T}$  is the full tree-shift over  $|\mathcal{A}| = k$  and each node has exactly  $d$  children. Then the entropy and hidden entropy of  $\mathcal{T}$  is  $\ln d$  and  $\ln k/(d - 1)$ , respectively.

Let  $\mathcal{A} = \{x^{(1)}, x^{(2)}, \dots, x^{(k)}\}$  be the (ordered) symbol set and  $X_{x^{(i)}}^{\mathcal{B}}$  denote the collection of trees  $t$  satisfying  $t_\epsilon = x^{(i)}$ , where  $1 \leq i \leq k$ . Given  $n \in \mathbb{N}$  with  $n \geq 2$ , let  $\mathcal{A}^n$  be an ordered set with respect to the lexicographic order. More explicitly, for any two  $n$ -words  $\mathbf{x} = x_1 x_2 \cdots x_n$  and  $\mathbf{y} = y_1 y_2 \cdots y_n$ , we say that  $\mathbf{x} < \mathbf{y}$ , if and only if there exists  $1 < r \leq n$  such that  $x_i = y_i$  for  $i \leq r-1$  and  $x_r < y_r$ , and  $\mathbf{x} = \mathbf{y}$  otherwise.

**Definition 1.3.** Let  $\mathcal{A} = \{x^{(1)}, \dots, x^{(k)}\}$  be the symbol set and suppose  $\mathcal{A}^d$  is an ordered set with respect to the lexicographic order,  $d \in \mathbb{N}$ .

- (1) Let  $F(x^{(1)}, \dots, x^{(k)}) = \sum_{\mathbf{x} \in \mathcal{A}^d} f_{\mathbf{x}} \mathbf{x}$  be a binary combination over  $\mathcal{A}^d$ , i.e.,  $f_{\mathbf{x}} \in \{0, 1\}$  for  $\mathbf{x} \in \mathcal{A}^d$ . The vector  $v_F = (f_{\mathbf{x}})_{\mathbf{x} \in \mathcal{A}^d} \in \mathbb{R}^{k^d}$  is called the *indicator vector* of  $F$ .
- (2) Suppose  $v_F$  and  $v_G$  are the indicator vectors of  $F = \sum_{\mathbf{x} \in \mathcal{A}^d} f_{\mathbf{x}} \mathbf{x}$  and  $G = \sum_{\mathbf{x} \in \mathcal{A}^d} g_{\mathbf{x}} \mathbf{x}$ , respectively. We say that  $v_F$  *dominates*  $v_G$ , denoted by  $v_F \geq v_G$ , if  $f_{\mathbf{x}} \geq g_{\mathbf{x}}$  for all  $\mathbf{x} \in \mathcal{A}^d$ . In other words,  $v_F - v_G$  is a nonnegative vector.

We remark that, since  $\mathcal{A}^d$  is an ordered set with respect to the lexicographic order, each binary combination  $F(x^{(1)}, \dots, x^{(k)}) = \sum_{\mathbf{x} \in \mathcal{A}^d} f_{\mathbf{x}} \mathbf{x}$  is also an ordered combination. For instance,  $v_{(x^{(1)})^d} = (1, 0, \dots, 0)$ , and  $v_F = e_{k^d}$  if  $F = \sum_{\mathbf{x} \in \mathcal{A}^d} \mathbf{x}$ , where  $e_{k^d}$  is the vector whose entries are all 1's.

**Definition 1.4.** A sequence  $\{a_n^{(1)}, \dots, a_n^{(k)}\}_{n \in \mathbb{N}}$  is defined by a *system of nonlinear recurrence equations* (SNRE) of degree  $(d, k)$  if

$$a_n^{(i)} = F^{(i)}(a_{n-1}^{(1)}, a_{n-1}^{(2)}, \dots, a_{n-1}^{(k)}) \quad \text{for } n \geq 2, 1 \leq i \leq k,$$

and  $a_1^{(i)} = a^{(i)}$  for  $1 \leq i \leq k$ , where  $F^{(1)}, \dots, F^{(k)}$  are binary combinations over  $\{a_{n-1}^{(1)}, a_{n-1}^{(2)}, \dots, a_{n-1}^{(k)}\}^d$ , respectively.

For the case where  $k = 2$ , set  $\mathcal{A} = \{x^{(1)}, x^{(2)}\} = \{1, 2\}$ ,  $a_n(\mathcal{B}) = |B_n(X_1^{\mathcal{B}})|$ ,  $b_n(\mathcal{B}) = |B_n(X_2^{\mathcal{B}})|$  and  $c_n(\mathcal{B}) = |B_n(X^{\mathcal{B}})|$ . We shall usually shorten  $a_n(\mathcal{B})$ ,  $b_n(\mathcal{B})$ , and  $c_n(\mathcal{B})$  to  $a_n$ ,  $b_n$ , and  $c_n$ , respectively, when  $\mathcal{B}$  is understood. For each  $n \in \mathbb{N}$ , let  $\Gamma_n : \{1, 2\} \rightarrow \{a_n, b_n\}$  be defined as  $\Gamma_n(1) = a_n$  and  $\Gamma_n(2) = b_n$ . We have the following theorem.

**Theorem 1.5** (SNRE for  $d = k = 2$ ). *Given a basic set  $\mathcal{B}$ ; then  $c_n = a_n + b_n$  and  $a_n$  and  $b_n$  satisfy the following SNRE of degree  $(2, 2)$ :*

$$\begin{cases} a_n = \sum_{(1,j,k) \in \mathcal{B}} \Gamma_{n-1}(j) \Gamma_{n-1}(k), \\ b_n = \sum_{(2,j,k) \in \mathcal{B}} \Gamma_{n-1}(j) \Gamma_{n-1}(k), \\ a_2 = |B_2(X_1^{\mathcal{B}})|, b_2 = |B_2(X_2^{\mathcal{B}})|. \end{cases}$$

*Proof.* It can be easily checked that  $c_n = a_n + b_n$  and, if  $u = (i, j, k) \in \mathcal{B}$ , then the numbers of  $n$ -blocks that has  $u$  rooted at  $\epsilon$  is  $\Gamma_{n-1}(j) \Gamma_{n-1}(k)$ . Therefore, we have

$$a_n = \sum_{(1,j,k) \in \mathcal{B}} \Gamma_{n-1}(j) \Gamma_{n-1}(k), b_n = \sum_{(2,j,k) \in \mathcal{B}} \Gamma_{n-1}(j) \Gamma_{n-1}(k), \quad n \geq 2,$$

with the initial condition  $a_2 = |B_2(X_1^{\mathcal{B}})|$  and  $b_2 = |B_2(X_2^{\mathcal{B}})|$ . This completes the proof.  $\square$

Theorem 1.5 indicates that each basic set of 2-blocks associates with an SNRE of degree  $(2, 2)$ . In general, Theorem 1.6, which is an extension of Theorem 1.5, reveals that the calculation of the topological entropy of a tree-shift of finite type is equivalent to solving a system of nonlinear recurrence equations.

**Theorem 1.6.** *The topological entropy of a tree-shift of finite type is realized as a system of nonlinear recurrence equations of degree  $(d, k)$  for some  $d, k$ . Conversely, every system of nonlinear recurrence equations of degree  $(d, k)$  corresponds to the topological entropy of some tree-shifts of finite type.*

*Proof.* Suppose  $X$  is a tree-shift of finite type. Without loss of generality, we may assume that  $X = X_{\mathcal{F}}$  is a Markov tree-shift over  $\mathcal{A} = \{1, 2, \dots, k\}$  with exactly  $d$  children for each node of  $t \in X$ . Recall that  $X$  is called a Markov tree-shift if  $\mathcal{F}$  consists of 2-blocks.

For  $1 \leq i \leq k$ , let  $a_n^{(i)} = |B_n(X_i)|$  be the cardinal number of  $n$ -blocks of  $X_i$  for  $n \in \mathbb{N}$ . Then  $|B_n(X)| = \sum_{i=1}^k a_n^{(i)}$ . Write  $u \in B_2(X)$  as  $u = (u_\epsilon, u_0, \dots, u_{d-1})$ . It is easily seen that

$$a_n^{(i)} = \sum_{u \in B_2(X_i)} a_{n-1}^{(u_0)} a_{n-1}^{(u_1)} \dots a_{n-1}^{(u_{d-1})}, \quad \text{for } 1 \leq i \leq k, n \geq 3.$$

Define

$$F^{(i)}(\alpha_1, \alpha_2, \dots, \alpha_k) = \sum_{u \in B_2(X_i)} \alpha_{n-1}^{(u_0)} \alpha_{n-1}^{(u_1)} \dots \alpha_{n-1}^{(u_{d-1})}.$$



It follows that the sequence  $\{a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(k)}\}_{n \geq 2}$  satisfies the SNRE

$$\begin{cases} a_n^{(i)} = F^{(i)}(a_{n-1}^{(1)}, a_{n-1}^{(2)}, \dots, a_{n-1}^{(k)}), & n \geq 3; \\ a_2^{(i)} = |B_2(X_i)|, & 1 \leq i \leq k. \end{cases}$$

Hence, the topological entropy of  $X$  is realized by the above SNRE.

Conversely, suppose  $\{a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(k)}\}_{n \in \mathbb{N}}$  satisfies the following SNRE of degree  $(d, k)$ :

$$\begin{cases} a_n^{(i)} = f_i(a_{n-1}^{(1)}, a_{n-1}^{(2)}, \dots, a_{n-1}^{(k)}), & n \geq 2; \\ a_1^{(i)} = a_i, & 1 \leq i \leq k. \end{cases}$$

Set

$$\mathcal{B}_i = \{(i, p_1, \dots, p_d) : a_{\mathbf{x}} \neq 0, \text{ where } f_i = \sum a_{\mathbf{x}} \mathbf{x}, \mathbf{x} = \alpha_{p_1} \dots \alpha_{p_d} \in \Lambda^d\},$$

where  $\Lambda = \{\alpha_1, \dots, \alpha_k\}$ . Let  $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$  and let  $\mathcal{F} = \mathcal{A}^2 \setminus \mathcal{B}$ , herein  $\mathcal{A} = \{1, \dots, k\}$ .

Similar to the discussion above, it can be verified without difficulty that  $X = X_{\mathcal{F}}$  is a Markov tree-shift and  $B_n(X_i) = a_{n-1}^{(i)}$  for  $n \geq 3$  and  $1 \leq i \leq k$ .

This completes the proof.  $\square$

## 2. ENTROPY FOR $d = k = 2$

Let  $d = k = 2$  and let  $\mathcal{B}$  be a basic set which associated with an SNRE (see Theorem 1.5). In this case, we have a pair of indicator vectors  $(v_F, v_G)$  (see Definition 1.3), where

$$\begin{aligned} F(a_{n-1}, b_{n-1}) &= \sum_{(1,j,k) \in \mathcal{B}} \Gamma_{n-1}(j) \Gamma_{n-1}(k), \\ G(a_{n-1}, b_{n-1}) &= \sum_{(2,j,k) \in \mathcal{B}} \Gamma_{n-1}(j) \Gamma_{n-1}(k), \end{aligned}$$

and  $\Gamma_n(1) = a_n, \Gamma_n(2) = b_n$ , respectively. An SNRE is of the *dominant-type* if either  $v_F$  dominates  $v_G$  or  $v_G$  dominates  $v_F$ . Finally we define  $n_* := |\{i : (v_*)_i \neq 0\}|$ , where  $*$  stands for  $F$  or  $G$  and  $(v)_i$  is the  $i$ -coordinate of  $v$ . Such number counts the number of nonzero entries in  $v_F$  and  $v_G$ , respectively.

**2.1. Classical results.** In [2], Aho and Sloane show that a sequence of natural numbers  $\{x_i\}_{i \geq 1}$  satisfying a nonlinear recurrence of the form  $x_{n+1} = x_n^2 + g_n$ , with  $|g_n| < \frac{1}{4}x_n$  for  $n \geq n_0$  has doubly exponential form of  $x_n \approx k^{2^n}$  for some  $k$ . The constant  $k$  is unknown in general. Ionascu and Stanica [17] extend Aho and Sloane's result and formulate  $x_n$  explicitly for some  $g_n$ . Many researchers have devoted to the study of nonlinear recurrence equations and only a few results are obtained (cf. [14]). Theorem 1.6 infers

the corresponding between SNRE and tree-shifts of finite type, which reveals the difficulty of the computation of entropy.

**Lemma 2.1.** *Suppose, for  $n \geq 1$ ,  $x_n$  satisfies a nonlinear recurrence equation*

$$\begin{cases} x_{n+1} = x_n^2 + |g_n|, \\ |g_n| \leq x_n, \\ x_1 = x^1. \end{cases}$$

*Then*

$$\lim_{n \rightarrow \infty} \frac{\ln^2 x_n}{n} = \ln 2.$$

**2.2. Complete characterization of entropy for  $d = k = 2$ .** This subsection elaborates the entropy of tree-shifts of finite type for the case where  $d = k = 2$ . Further investigation is discussed in the upcoming section.

**Lemma 2.2.** *If an SNRE corresponding to a given  $\mathcal{B}$  is of the dominant-type and satisfies one of the following conditions:*

- (1)  $(v_F)_1 \neq 0$  and  $n_F \geq 2$ ,
- (2)  $(v_F)_1 = 0$ , and  $n_F$  and  $n_G \geq 2$ ,

*then  $h(\mathcal{B}) = \ln 2$ .*

*Proof.* Without loss of generality, we assume  $v_F \geq v_G$ . It follows that  $a_n \geq b_n$  for all  $n \geq 2$  and

$$h(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln^2 c_n}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 (a_n + b_n)}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 a_n}{n}.$$

Thus, it suffices to compute  $\lim_{n \rightarrow \infty} \frac{\ln^2 a_n}{n}$ . We divide the proof into two cases.

**1.**  $(v_F)_1 \neq 0$ . A similar discussion to the proof of Lemma 2.1 indicates that  $h(\mathcal{B}) = \ln 2$ .

**2.**  $(v_F)_1 = 0$ . We divide this case into several sub-cases.

**2-a.**  $v_F = (0, 1, 1, 1)$ . In this case, since  $n_G \geq 2$ , it is seen that  $v_G = (0, 1, 1, 0)$ ,  $(0, 0, 1, 1)$ , or  $(0, 1, 0, 1)$ . For the case where  $v_G = (0, 1, 1, 0)$ , we define a SNRE as follows.

$$\begin{cases} d_n = d_{n-1}e_{n-1} + e_{n-1}d_{n-1}, \\ e_n = d_{n-1}e_{n-1} + e_{n-1}d_{n-1}, \\ d_2 = e_2 = 2. \end{cases}$$

Then we have  $d_n = e_n$ ,  $a_n \geq d_n$ , and  $b_n \geq e_n$  for  $n \geq 2$ , which yields that

$$\ln 2 \geq h(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln^2 c_n}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 a_n}{n} \geq \lim_{n \rightarrow \infty} \frac{\ln^2 d_n}{n} = \ln 2.$$

Hence  $h(\mathcal{B}) = \ln 2$ . Suppose  $v_G = (0, 0, 1, 1)$ . Define a SNRE as follows.

$$\begin{cases} d_n = d_{n-1}e_{n-1}, \\ e_n = e_{n-1}d_{n-1} + e_{n-1}^2, \\ d_2 = 1 \text{ and } e_2 = 2. \end{cases}$$

Thus we have  $a_n \geq d_n$ ,  $b_n \geq e_n \geq d_n$  for  $n \geq 2$ . Since  $v_F > v_G$ , it follows that

$$\begin{aligned} \ln 2 \geq h(\mathcal{B}) &= \lim_{n \rightarrow \infty} \frac{\ln^2 c_n}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 a_n}{n} \geq \lim_{n \rightarrow \infty} \frac{\ln^2 d_n}{n} \\ &\geq \lim_{n \rightarrow \infty} \frac{\ln^2 e_{n-1}}{n} = \ln 2. \end{aligned}$$

Note that the same proof of case **1** works for  $\lim_{n \rightarrow \infty} \frac{\ln^2 e_{n-1}}{n} = \ln 2$ . The case where  $v_G = (0, 1, 0, 1)$  can be demonstrated analogously. This completes the proof.

**2-b.**  $v_F = (0, 0, 1, 1)$  or  $(0, 1, 0, 1)$ . We deal with the case  $v_F = (0, 0, 1, 1)$ . The other can be indicated similarly. Since  $n_G \geq 2$ ,  $v_G$  has only one possibility  $v_G = (0, 0, 1, 1)$ . It follows immediately that  $a_n = b_n$  for all  $n$  and the SNRE is transformed into  $a_n = 2a_{n-1}^2$ ,  $a_2 \geq 2$ . Thus  $h(\mathcal{B}) = \ln 2$ .

**2-c.**  $v_F = (0, 1, 1, 0)$ . In this case,  $v_G = (0, 1, 1, 0)$ . Applying the same argument given in **2-b** yields that  $h(\mathcal{B}) = \ln 2$ . The proof is thus complete.  $\square$

**Remark 2.3.** If we soften the condition (ii) in Lemma 2.2 to  $n_G < 2$ , then, in the proof of **2-a**, we only need to deal with  $v_G = (0, 0, 0, 1)$  (the cases where  $v_G = (0, 1, 0, 0)$  and  $v_G = (0, 0, 1, 0)$  are similar). In this case, it is seen that  $b_n = 1$  for all  $n \geq 2$  and the equation becomes  $a_n = 2a_{n-1} + a_{n-1}^2$ . In this case we still have  $h(\mathcal{B}) = \ln 2$ . In **2-b**, it remains to elaborate the case where  $v_F = (0, 0, 1, 1)$ . It follows that  $v_G = (0, 0, 1, 0)$  or  $v_G = (0, 0, 0, 1)$ . If  $v_G = (0, 0, 1, 0)$ , then  $b_n \geq a_{n-1}$  for all  $n \geq 2$ . Define the a SNRE as follows.

$$\begin{cases} d_n = d_{n-1}e_{n-1}, \\ e_n = d_{n-1}e_{n-1}, \\ d_2 = 2, e_2 = 1. \end{cases}$$

It is evident that  $a_n \geq d_n$  and  $b_n = e_n$  for  $n \geq 2$ . Thus we have  $d_n = d_{n-1}^2$  with  $d_2 = 2$ . A straightforward examination asserts that  $h(\mathcal{B}) = \ln 2$ . Finally,  $v_F = (0, 1, 1, 0)$  infers that  $v_G = (0, 1, 0, 0)$  or  $v_G = (0, 0, 1, 0)$ . Both cases can be analyzed analogously as above.

Corollary 2.4 is generalized from Lemma 2.2 based on the discussion in Remark 2.3.

**Corollary 2.4.** *If the SNRE corresponding to a given  $\mathcal{B}$  is of the dominant-type with  $n_F \geq 2$ , then  $h(\mathcal{B}) = \ln 2$ .*

**Example 2.5.** Suppose the basic set  $\mathcal{B}$  is given by

$$\mathcal{B} = \{(1, 1, 1), (1, 2, 2), (2, 2, 2)\}.$$

It follows that the indicator vectors of the corresponding SNRE are  $v_F = (1, 0, 0, 1)$  and  $v_G = (0, 0, 0, 1)$ , and the SNRE is of the dominant-type. Corollary 2.4 elaborates that  $h(\mathcal{B}) = \ln 2$ .

The following corollary comes immediately from Corollary 2.4, with the proof omitted.

**Corollary 2.6.** *If the SNRE corresponding to a given  $\mathcal{B}$  is of the form  $v_F = (1, *, *, *)$  with  $n_F \geq 2$  or  $v_G = (*, *, *, 1)$  with  $n_G \geq 2$ . Then  $h(\mathcal{B}) = \ln 2$ .*

Corollary 2.6 demonstrates that positive entropy derives from  $F$  (resp.  $G$ ) containing at least two terms and one of which is  $a_{n-1}^2$  (resp.  $b_{n-1}^2$ ). To illustrate the entropy of tree-shifts of finite type completely, we introduce another type of SNRE. Two vectors  $v$  and  $w$  are *complementary* if  $v + w$  dominates  $e_4$ , where  $e_4 = (1, 1, 1, 1)$ . An SNRE obtained from a given basic set  $\mathcal{B}$  is of the *complementary-type* if the corresponding indicator vectors  $v_F$  and  $v_G$  are complementary.

**Lemma 2.7.** *Suppose a basic set  $\mathcal{B}$  is given. If the corresponding SNRE is of complementary-type, then  $h(\mathcal{B}) = \ln 2$ .*

*Proof.* If the SNRE is of the complementary-type. Then we have

$$c_n = a_n + b_n \geq a_{n-1}^2 + 2a_{n-1}b_{n-1} + b_{n-1}^2 = c_{n-1}^2$$

with  $c_2 = a_2 + b_2 \geq 4$ . Suppose the sequence  $\{d_n\}$  satisfies  $d_n = (d_{n-1})^2$  and  $d_2 = 4$ . It can be easily checked that  $c_n \geq d_n$  for all  $n \geq 2$ . Hence, we have

$$h(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln^2 c_n}{n} \geq \lim_{n \rightarrow \infty} \frac{\ln^2 d_n}{n} \geq \ln 2.$$

This completes the proof.  $\square$

**Corollary 2.8.** *If the SNRE corresponding to a given  $\mathcal{B}$  satisfies  $v_F + v_G \geq (1, 0, 1, 1)$  or  $(1, 1, 0, 1)$ , then  $h(\mathcal{B}) = \ln 2$ .*

*Proof.* Notably,

$$c_n = a_n + b_n \geq a_{n-1}^2 + a_{n-1}b_{n-1} + b_{n-1}^2 \geq \frac{1}{2}c_{n-1}^2.$$

Suppose  $\{d_n\}$  satisfies  $d_n = \frac{1}{2}d_{n-1}^2$  and  $d_2 \geq 3$ . Then we have  $c_n \geq d_n$ , and hence  $h(\mathcal{B}) = \ln 2$ . This completes the proof.  $\square$

**Example 2.9.** Suppose the basic set  $\mathcal{B}$  is given as

$$\mathcal{B} = \{(1, 1, 1), (1, 1, 2), (2, 2, 1), (2, 2, 2)\}.$$

It is seen that the indicator vectors of the corresponding SNRE are  $v_F = (1, 1, 0, 0)$  and  $v_G = (0, 0, 1, 1)$ . Hence the SNRE is of complementary-type. Lemma 2.7 asserts that the entropy of tree-shift  $X^{\mathcal{B}}$  is  $h(\mathcal{B}) = \ln 2$ .

With the discussion above, we are in the position of demonstrating the main result of this section.

**Theorem 2.10.** *Suppose  $d = k = 2$ . Then the entropy of the tree-shift of finite type generated from a basic set  $\mathcal{B}$  is either  $h(\mathcal{B}) = 0$  or  $h(\mathcal{B}) = \ln 2$ .*

*Proof.* The demonstration is implemented in cases.

**I.**  $v_F = (1, *, *, *)$  with  $n_F \geq 2$ . This case comes directly from Corollary 2.6.

**II.**  $v_F = (0, *, *, *)$  with  $n_F = 3$ , i.e.,  $v_F = (0, 1, 1, 1)$ . In this case, the investigation of  $v_G = (1, 0, 0, 0)$  and  $v_G = (0, *, *, *)$  is conducted by Lemma 2.7 and Corollary 2.4, respectively.

**III.**  $v_F = (0, *, *, *)$  with  $n_F = 2$ . The Pigeonhole Principle indicates that at least one of the center two entries of  $v_F$  is 1. The elucidation of  $v_F = (0, 1, 0, 1)$  can be treated similarly as the discussion of  $v_f = (0, 0, 1, 1)$ . The detailed implementation is divided into several sub-cases and is listed in the tables below.

**III-1.**  $n_G \geq 3$ .

	$v_F = (0, 1, 1, 0)$	$v_F = (0, 0, 1, 1)$
$v_G = (1, 1, 1, 1)$	Lemma 2.2	Lemma 2.2
$v_G = (1, 0, 1, 1)$	Corollary 2.6	Corollary 2.6
$v_G = (1, 1, 0, 1)$	Corollary 2.6	Corollary 2.6
$v_G = (0, 1, 1, 1)$	Corollary 2.6	Corollary 2.6
$v_G = (1, 1, 1, 0)$	Lemma 2.2	Lemma 2.7

**III-2.**  $n_G = 2$

	$v_F = (0, 1, 1, 0)$	$v_F = (0, 0, 1, 1)$
$v_G = (1, 0, 0, 1)$	Lemma 2.7	Corollary 2.8
$v_G = (1, 0, 1, 0)$	(1)	Corollary 2.8
$v_G = (1, 1, 0, 0)$	(2)	Lemma 2.7
$v_G = (0, 1, 0, 1)$	Corollary 2.6	Corollary 2.6
$v_G = (0, 1, 1, 0)$	$a_n = b_n$ for all $n \geq 2$	(3)
$v_G = (0, 0, 1, 1)$	Corollary 2.6	Corollary 2.6

**III-3.**  $n_G = 1$

	$v_F = (0, 1, 1, 0)$	$v_F = (0, 0, 1, 1)$
$v_G = (1, 0, 0, 0)$	(4)	Corollary 2.8
$v_G = (0, 1, 0, 0)$	Lemma 2.2	(5)
$v_G = (0, 0, 1, 0)$	Lemma 2.2	Lemma 2.2
$v_G = (0, 0, 0, 1)$	(6)	Lemma 2.2

It only remains to deal with Cases (1)-(6). Notably, Cases (1)-(3) can be treated similarly. The equation for Case (1) is

$$\begin{cases} a_n = 2a_{n-1}b_{n-1}, \\ b_n = a_{n-1}^2 + a_{n-1}b_{n-1}, \\ a_2 = b_2 = 2. \end{cases}$$

We claim that  $a_n = b_n$  for all  $n \geq 2$ . Observe that  $a_2 = b_2$ . Suppose  $a_k = b_k$  for some  $2 \leq k \in \mathbb{N}$ . Then

$$a_{k+1} = 2a_k b_k = a_k b_k + a_k b_k = a_k^2 + a_k b_k = b_{k+1}.$$

The claim follows by mathematical induction. Thus the system is reduced to solving  $a_n$  which satisfies  $a_n = 2a_{n-1}^2$  and  $a_2 = 2$ . Hence we have  $h(\mathcal{B}) = \ln 2$ .

In Case (4), we have the corresponding SNRE as follows.

$$\begin{cases} a_n = 2a_{n-1}b_{n-1}, \\ b_n = a_{n-1}^2, \\ a_2 = 2, \quad b_2 = 1. \end{cases}$$

This is equivalent to solving  $a_n = 2a_{n-1}a_{n-2}^2$  with  $a_2 = 2$ . Suppose  $\{d_n\}$  is defined as

$$d_n = d_{n-1}d_{n-2}^2, \quad d_2 = 2.$$

It is seen that  $a_n \geq d_n$  for all  $n \geq 2$ . Let  $d_n = \alpha^{\beta^n}$  we then have  $\beta^2 - \beta - 2 = 0$ , which implies  $\beta = 2$ , i.e.,  $h(\mathcal{B}) = \ln 2$ . This completes the discussion of Case (4).

In Case (5), we have the following SNRE

$$\begin{cases} a_n = a_{n-1}b_{n-1} + b_{n-1}^2, \\ b_n = a_{n-1}b_{n-1}, \\ a_2 = 2, \quad b_2 = 1. \end{cases}$$

Since  $a_n \geq b_n$  for all  $n \geq 2$ . Using the same method of the proof in Lemma 2.2 and Remark 2.3 we have  $h(\mathcal{B}) = \ln 2$ . Finally, the SNRE of Case (6) is as follows.

$$\begin{cases} a_n = 2a_{n-1}b_{n-1}, \\ b_n = b_{n-1}^2, \\ a_2 = 2, \quad b_2 = 1. \end{cases}$$

It is evident that  $b_n = 1$  for  $n \geq 2$ . Thus we have  $a_n = 2a_{n-1}$  for  $a_2 = 2$ . It is a simple matter to see that  $h(\mathcal{B}) = 0$ . The proof is thus complete.  $\square$

3. ENTROPY FOR  $d, k \geq 2$ 

This section investigates the entropy of tree-shifts of finite type defined on infinite trees, in which each node has exactly  $d$  children for  $d \geq 2$ , with labels in  $\mathcal{A} = \{1, \dots, k\}$  of cardinality  $k \geq 2$ . The corresponding SNRE becomes

$$(3) \quad \begin{cases} a_n^{(1)} = F^{(1)}(a_{n-1}^{(1)}, \dots, a_{n-1}^{(k)}), \\ \vdots \\ a_n^{(k)} = F^{(k)}(a_{n-1}^{(1)}, \dots, a_{n-1}^{(k)}), \\ a_2^{(i)} = a_i = |B_2(X_i^{\mathcal{B}})| \text{ for } i = 1, \dots, k, \end{cases}$$

where  $F^{(i)}$  is a binary combination over  $\{a_{n-1}^{(1)}, \dots, a_{n-1}^{(k)}\}^d$  for  $1 \leq i \leq k$ .

**3.1. Realization theorem.** The last section indicates that, for the case where  $d = k = 2$ , the entropy of a tree-shift of finite type is either 0 or  $\ln 2$ , provided the forbidden set consists of 2-blocks. Recall that the class of such tree-shifts of finite type is also known as Markov tree-shifts [4]. Roughly speaking, a binary Markov tree-shift labeled by two symbols is of either the simplest or the most complicated dynamical behavior. There is no Markov tree-shift in between. It is then natural to ask if this is the case in general.

In this section, we demonstrate that tree-shifts are capable of rich dynamical behavior by showing the entropy set of tree-shifts of finite type covers a large number of reals which include the multinacci numbers.

**Theorem 3.1** (Realization Theorem). *Let  $\rho$  be the maximal root of  $x^n - \sum_{i=1}^{n-1} k_i x^i = 0$ , where  $k_i \in \mathbb{N} \cup \{0\}$  for  $i = 1, \dots, n-1$ . Then there exist  $d, k \geq 2$  and  $\mathcal{B}$  such that  $h(\mathcal{B}) = \ln \rho$ .*

*Proof.* Assume that  $\rho$  is the maximal root of

$$(4) \quad F(x) = x^p - k_1 x^{p_1} - k_2 x^{p_2} - \dots - k_{l-1} x^{p_{l-1}} - k_l x^{p_l}, \quad k_i \in \mathbb{N}, \quad \forall 1 \leq i \leq l,$$

where  $p_l = 0$ . Let  $q_i = p - p_i$  for  $i = 1, \dots, l$ . Thus  $\rho$  satisfies the equation

$$(5) \quad 1 = \sum_{j=1}^l k_j x^{-q_j}.$$

The proof is divided into two parts.

**1.**  $p_1 < p - 1$ . Introduce the symbol set  $\bigcup_{j=1}^l \bigcup_{i=0}^{q_j-1} \{a^{(j,i)}\} \cup \{a^{(0)}, b\}$  and denote  $a^{(j)} = a^{(j,0)}$  for all  $1 \leq j \leq l$ . Define  $d = \sum_{i=1}^l k_i$  and  $k = 2 + \sum_{j=1}^l q_j$ , we

construct the SNRE according to  $d$  and  $k$  as follows.

$$(6) \quad \begin{cases} a_n^{(0)} = 2 \left( a_{n-1}^{(1)} \right)^{k_1} \left( a_{n-1}^{(2)} \right)^{k_2} \left( a_{n-1}^{(3)} \right)^{k_3} \cdots \left( a_{n-1}^{(l)} \right)^{k_l}, \\ a_n^{(j,r)} = a_{n-1}^{(j,r+1)} b_{n-1}^{d-1}, \text{ for all } 1 \leq j \leq l, 0 \leq r \leq q_j - 2, \\ a_n^{(j,q_j-1)} = a_{n-1}^{(0)} b_{n-1}^{d-1}, \\ b_n = b_{n-1}^d, \\ a_2^{(0)} = 2, b_2 = a_2^{(j,r)} = 1, \forall 1 \leq j \leq l \text{ and } 0 \leq r \leq q_j - 1. \end{cases}$$

We claim that  $h(\mathcal{B}) = \ln \rho$ . Since for  $1 \leq j \leq l$  and  $1 \leq r \leq q_j - 2$ ,

$$(7) \quad \begin{aligned} a_n^{(j)} &= a_n^{(j,0)} = a_{n-1}^{(j,1)} b_{n-1}^{d-1} = a_{n-2}^{(j,2)} b_{n-2}^{d-1} b_{n-1}^{d-1} \\ &= \cdots = a_{n-r}^{(j,r)} (b_{n-r} \cdots b_{n-1})^{d-1} \\ &= a_{n-(q_j-1)}^{(j,q_j-1)} (b_{n-(q_j-1)} \cdots b_{n-1})^{d-1}. \end{aligned}$$

Combining (7) with the fact that  $a_n^{(j,q_j-1)} = a_{n-1}^{(0)} b_{n-1}^{d-1}$  we obtain

$$(8) \quad a_n^{(j)} = a_{n-q_j}^{(0)} (b_{n-q_j} \cdots b_{n-1})^{d-1} \text{ for all } 1 \leq j \leq l.$$

Substituting (8) in the first equation of (6) yields

$$(9) \quad a_n^{(0)} = 2 \prod_{j=1}^l \left( a_{n-q_j}^{(0)} \right)^{k_j} (b_{n-q_j} \cdots b_{n-1})^{k_j(d-1)}.$$

One can easily check that  $b_n = 1$  for all  $n \geq 1$ , thus we can rewrite (9) as follows.

$$a_n^{(0)} = 2 \prod_{j=1}^l \left( a_{n-q_j}^{(0)} \right)^{k_j} \text{ with } a_2^{(0)} = 2.$$

Taking  $x_n = \ln a_n^{(0)}$  becomes

$$x_n = \ln 2 + \sum_{j=1}^l k_j x_{n-q_j}.$$

Let  $x_n = \alpha^n$ . We have the following equation

$$(10) \quad \alpha^n = \ln 2 + \sum_{j=1}^l k_j \alpha^{n-q_j}.$$

Since  $\alpha > 1$ , dividing (10) by  $\alpha^n$  and letting  $n$  approach to infinity infers

$$1 = \sum_{j=1}^l k_j \alpha^{-q_j}.$$

That is,  $\alpha = h(\mathcal{B})$  satisfies (5).

**2.**  $p_1 = p - 1$ . The same proof works when we modify the first equation of (6) as follows.

$$a_n^{(0)} = 2 \left( a_{n-1}^{(0)} \right)^{k_1} \left( a_{n-1}^{(1)} \right)^{k_2} \left( a_{n-1}^{(2)} \right)^{k_3} \cdots \left( a_{n-1}^{(l)} \right)^{k_l}.$$



This completes the proof.  $\square$

It is worth pointing out that the *multinacci number* of order  $n \in \mathbb{N} \setminus \{1\}$  is the real number  $\gamma_n \in (1, 2)$  which is the unique positive real solution of the equation  $1 = x^{-1} + x^{-2} + \dots + x^{-n}$ . It follows immediately from Theorem 3.1 that the multinacci numbers of all order can be realized by tree-shifts of finite type. The smallest multinacci number is the golden ratio  $(1 + \sqrt{5})/2$ . The following example presents a tree-shift of finite type which realizes such number.

**Example 3.2.** Let  $g = (1 + \sqrt{5})/2$  be the maximal root of the polynomial

$$(11) \quad x^2 - x - 1 = 0.$$

In this case we have  $p_1 = p - 1$  (defined in the proof of Theorem 3.1). Construct the SNRE as follows.

$$\begin{cases} a_n^{(0)} = 2a_{n-1}^{(0)}a_{n-1}^{(1)}, \\ a_n^{(1)} = a_{n-1}^{(0)}b_{n-1}, \\ b_n = b_{n-1}^2, \\ a_2^{(1)} = 2, \quad b_2 = a_2^{(2)} = 1. \end{cases}$$

Theorem 3.1 shows that  $h(\mathcal{B}) = \ln g$ . In this case, the desired pair of  $d$  and  $k$  are  $d = 2$  and  $k = 3$ .

**3.2. Computation of entropy for general  $d$  and  $k$ .** For SNRE (3) and  $1 \leq i \leq k$ , we associate an indicator vector  $v^{(i)} := v_{F(i)}$  as defined earlier. If there exists an  $l$  such that  $v^{(l)} \geq v^{(j)}$  for all  $1 \leq j \leq k$ , then we call  $v^{(l)}$  a *dominant vector*. We say that SNRE (3) is of the *dominant-type* if there exists a dominant vector  $v \in \{v^{(i)}\}_{i=1}^k$ .

**Example 3.3.** Let  $d = 2$  and  $k = 3$ . Given  $\mathcal{B}$  and suppose the corresponding SNRE is of the form.

$$(12) \quad \begin{cases} a_n^{(1)} = \left(a_{n-1}^{(1)}\right)^2 + a_{n-1}^{(1)}a_{n-1}^{(2)} + a_{n-1}^{(1)}a_{n-1}^{(3)} = F^{(1)}, \\ a_n^{(2)} = a_{n-1}^{(1)}a_{n-1}^{(2)} + a_{n-1}^{(1)}a_{n-1}^{(3)} = F^{(2)} \\ a_n^{(3)} = a_{n-1}^{(1)}a_{n-1}^{(3)} = F^{(3)} \\ a_2^{(1)} = 3, \quad a_2^{(2)} = 2 \text{ and } a_2^{(3)} = 1. \end{cases}$$

Then one can easily check that  $v^{(1)} = (1, 1, 1, 0, 0, 0, 0, 0, 0)$  is a dominant vector and the SNRE (12) is of the dominant-type.

**Proposition 3.4.** *Given  $\mathcal{B}$ , if the corresponding SNRE (3) is of the dominant-type with  $v^{(l)}$  being a dominant vector. Then  $h(\mathcal{B}) = \ln h$  or 0, where  $h$  is*

the maximal degree of  $a_{n-1}^{(l)}$  in  $F^{(l)}$ . Furthermore, if  $a_{n-1}^{(l)}$  has degree  $d$  in  $F^{(l)}$ , then  $h(\mathcal{B}) = \ln d$ .

*Proof.* There is no loss of generality in assuming  $l = 1$  and suppose the maximal degree of  $a_{n-1}^{(1)}$  in  $F^{(1)}$  is  $h$ . Following the same method as the proof of Lemma 2.2 we have  $h(\mathcal{B}) \geq \ln h$ . Thus it suffices to show that  $h(\mathcal{B}) \leq \ln h$ . Since  $v^{(1)}$  is dominant, we have  $a_n^{(1)} \geq a_n^{(i)}$  for  $i = 2, \dots, k$  and  $n \geq 2$ . Thus  $\sum_{i=1}^k a_n^{(i)} \leq k a_n^{(1)}$  and

$$h(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln^2 \sum_{i=1}^k a_n^{(i)}}{n} \leq \lim_{n \rightarrow \infty} \frac{\ln^2 a_n^{(1)}}{n} = \ln h.$$

The last equality follows from the same argument to the proof of Lemma 2.1, and the proof is complete.  $\square$

It is remarkable that Corollary 2.6 is a special case of Proposition 3.4 and it is a simple matter to see that  $h(\mathcal{B}) = \ln 2$  in Example 3.3.

In the following, we introduce another methodology for computing entropy, which shows that the symmetry of basic sets implies full entropy. Suppose  $\mathcal{B}$  is a basic set consisting of two-blocks. In this case, each  $u \in \mathcal{B}$  can be written as  $u = (u_\epsilon, u_0, u_1, \dots, u_{d-1})$ . Define a *projection map*  $\pi$  as

$$\pi(u) = (u_0, u_1, \dots, u_{d-1}) \quad \text{for all } u \in \mathcal{B}.$$

For instance, if  $u = (1, 2, 2, 1, 2, 1)$  is a two-block, then  $\pi(u) = (2, 2, 1, 2, 1)$ .

**Definition 3.5** (Symmetric basic sets). Given a basic set  $\mathcal{B}$ , we decompose it into  $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}^{(i)}$ , where

$$\mathcal{B}^{(i)} = \{\omega : \omega_\epsilon \text{ is labeled by } i\} \text{ for } 1 \leq i \leq k.$$

Here we say the corresponding tree-shift  $X^{\mathcal{B}}$  is *symmetric* if  $\pi(\mathcal{B}^{(i)}) = \pi(\mathcal{B}^{(j)})$  for all  $i \neq j$ . In this case we call the corresponding SNRE (3) is of the *symmetric-type*.

Suppose  $v^{(i)}$  is an indicator vector, we denote by  $n^{(i)} = \left| \left\{ j : (v^{(i)})_j \neq 0 \right\} \right|$  the number of the non-zero coordinates of  $v^{(i)}$ . The following result shows that the entropy of a symmetric tree-shift is  $\ln d$ .

**Proposition 3.6.** *If an SNRE is of symmetric-type with  $n^{(1)} \geq 2$ . Then  $h(\mathcal{B}) = \ln d$ .*

*Proof.* Symmetric-type property implies  $a_n^{(i)} = a_n^{(j)}$  and  $n^{(i)} = n^{(j)}$  for all  $1 \leq i, j \leq k$  and  $n \geq 2$ . Then the corresponding SNRE (3) can be reduced to a single equation  $a_n^{(1)} = n^{(1)} \left( a_{n-1}^{(1)} \right)^d$  with  $n^{(1)} \geq 2$ . Hence we have  $h(\mathcal{B}) = \ln d$ , which proves the theorem.  $\square$

**Example 3.7.** Suppose a Markov tree-shift  $X^{\mathcal{B}}$  is generated by the basic set  $\mathcal{B} = \{(i, 1, 1, 2), (i, 1, 2, 1), (i, 2, 1, 1), (i, 2, 1, 2), (i, 2, 2, 2) : i = 1, 2\}$ . Then the indicator vectors of the corresponding SNRE are

$$v^{(1)} = (0, 1, 1, 0, 1, 1, 0, 1) = v^{(2)}.$$

Hence the SNRE is of the symmetric-type and  $n^{(1)} = 5 = n^{(2)}$ . Proposition 3.6 indicates that the entropy of  $X^{\mathcal{B}}$  is  $h(\mathcal{B}) = \ln 3$ .

#### 4. ENTROPY WITH BOUNDARY CONDITIONS

Suppose  $X$  is a tree-shift. It is natural to ask whether the topological entropy of  $X$  is influenced by the constraint of boundary conditions, and under what conditions the topological entropy of  $X$  with boundary conditions is identical to the topological entropy of the original tree-shift  $X$ .

In this section, we consider three different types of boundary conditions: *Periodic*, *Dirichlet*, and *Neumann boundary conditions*. For a one-sided shift space  $X$  with finite lattice  $n \in \mathbb{N}$ , we mean that  $x = x_1 x_2 \dots x_n$  for each  $x \in X$ . Periodic boundary condition infers that the underlying lattice of  $X$  is treated as a circle; that is,  $x_n = x_1$  for all  $x \in X$ . Dirichlet boundary condition indicates that the terminal state is constant for each  $x \in X$ . More precisely,  $x_n = \kappa$  for all  $x \in X$ , where  $\kappa$  is given. Neumann boundary condition means there is zero flux in the system; namely,  $x_n = x_{n-1}$  for all  $x \in X$ .

For the case where  $X$  is a tree-shift and  $n \geq 2$ , the collection of  $n$ -blocks in  $X$  with the constraint of periodic, Dirichlet, and Neumann boundary conditions, denoted by  $B_n^P(X)$ ,  $B_n^{D_i}(X)$ , and  $B_n^N(X)$ , respectively, is defined as

$$(13) \quad B_n^P = \{u \in B_n(X) : u_w = u_{\epsilon} \text{ for } |w| = n-1\},$$

$$(14) \quad B_n^{D_i} = \{u \in B_n(X) : u_w = i \text{ for } |w| = n-1\}, \quad 1 \leq i \leq k,$$

$$(15) \quad B_n^N = \{u \in B_n(X) : u_w = u_{\hat{w}} \text{ for } |w| = n-1, \hat{w} = w_1 \dots w_{n-2}\},$$

respectively. Based on the definitions of  $B_n^P$ ,  $B_n^{D_i}$ , and  $B_n^N$ , the topological entropy of tree-shift  $X$  with boundary condition is defined as

$$h^{\iota}(\mathcal{B}) = h^{\iota}(X^{\mathcal{B}}) = \lim_{n \rightarrow \infty} \frac{\ln^2 |B_n^{\iota}(X^{\mathcal{B}})|}{n}$$

provided the limit exists, herein  $\iota = P, D_1, \dots, D_k, N$ .

In the rest of this section,  $X = X^{\mathcal{B}}$  is a binary Markov tree-shift over  $\mathcal{A} = \{1, 2\}$  unless otherwise stated. Namely, we focus on the case where  $d = k = 2$ .

Obviously,  $h^\iota(\mathcal{B}) \leq h(\mathcal{B})$  for  $\iota = P, D_1, D_2, N$ . Hence, the effect of boundary conditions is seen only in those tree-shifts carrying positive topological entropy.

**Theorem 4.1.** *Suppose  $X^\mathcal{B}$  is a Markov tree-shift and  $h(\mathcal{B}) > 0$ . Then  $h^N(\mathcal{B}) = h(\mathcal{B})$  if and only if either (1)  $\{(1, 1, 1), (2, 2, 2)\} \subseteq \mathcal{B}$  or (2)  $\mathcal{B}$  contains  $\{(1, i, i), (2, i, i)\}$  for some  $i = 1, 2$ .*

*Proof.* We start the proof with the “Only If” part. It is easily seen that  $\{(1, 1, 1), (2, 2, 2)\} \subseteq \mathcal{F}$  asserts that  $h^N(\mathcal{B}) = 0 < h(\mathcal{B})$ . Suppose, without loss of generality,  $(1, 1, 1) \in \mathcal{B}$  and  $(2, 2, 2) \in \mathcal{F}$ . If  $(2, 1, 1) \in \mathcal{F}$ , then

$$|B_n^N(X^\mathcal{B})| = |B_{n-1}(X_1^\mathcal{B})| = 1 \quad \text{for } n \geq 2.$$

This contradicts to  $h^N(\mathcal{B}) = h(\mathcal{B})$ . Hence,  $\{(1, 1, 1), (2, 1, 1)\}$  is a subset of  $\mathcal{B}$ .

For the “If” part, observe that  $|B_n^N(X^\mathcal{B})| = |B_{n-1}(X^\mathcal{B})|$  follows from the presumption  $\{(1, 1, 1), (2, 2, 2)\} \subseteq \mathcal{B}$ , and thus  $h^N(\mathcal{B}) = h(\mathcal{B})$ . Without loss of generality, assume that  $\{(1, 1, 1), (2, 1, 1)\} \subseteq \mathcal{B}$ . It comes immediately that

$$|B_n^N(X^\mathcal{B})| \geq |B_{n-1}(X_1^\mathcal{B})| = |B_{n-2}(X^\mathcal{B})|,$$

which leads to the desired conclusion.

This completes the proof.  $\square$

**Theorem 4.2.** *Suppose  $X^\mathcal{B}$  is a Markov tree-shift and  $h(\mathcal{B}) > 0$ . Then, for  $i = 1, 2$ ,  $h^{D_i}(\mathcal{B}) = h(\mathcal{B})$  if and only if either (1)  $\{(1, i, i), (2, i, i)\} \subseteq \mathcal{B}$  or (2)  $\{(\bar{i}, i, i), (1, \bar{i}, \bar{i}), (2, \bar{i}, \bar{i})\} \subseteq \mathcal{B}$ , where  $i + \bar{i} = 3$ .*

*Proof.* We address the elucidation for the case where  $i = 1$ . The other case can be done analogously.

Start with the “If” part. Note that

$$B_n^{D_1}(X^\mathcal{B}) = \{u \in B_n(X^\mathcal{B}) : u_w = 1 \text{ for } |w| = n - 1\}.$$

Suppose  $\{(1, 1, 1), (2, 1, 1)\} \subseteq \mathcal{B}$ . It follows immediately that

$$|B_n^{D_1}(X^\mathcal{B})| = |B_{n-1}(X^\mathcal{B})|.$$

Hence,

$$h^{D_1}(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln^2 |B_n^{D_1}(X^\mathcal{B})|}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 |B_{n-1}(X^\mathcal{B})|}{n} = h(\mathcal{B}).$$

On the other hand,  $\{(2, 1, 1), (1, 2, 2), (2, 2, 2)\} \subseteq \mathcal{B}$  infers that

$$|B_n^{D_1}(X^\mathcal{B})| = |B_{n-1}^{D_2}(X^\mathcal{B})| = |B_{n-2}(X^\mathcal{B})|,$$

which concludes the coincidence of  $h^{D_1}(\mathcal{B})$  and  $h(\mathcal{B})$ .

Conversely,  $\{(1, 1, 1), (2, 1, 1)\} \subseteq \mathcal{F}$  derives that  $B_n^{D_1}(X^{\mathcal{B}}) = \emptyset$  for all  $n \geq 2$ . Hence  $0 = h^{D_1}(\mathcal{B}) < h(\mathcal{B})$ . Without loss of generality, we assume that  $(1, 1, 1) \in \mathcal{B}$  while  $(2, 1, 1) \in \mathcal{F}$ . Observe that

$$|B_n^{D_1}(X^{\mathcal{B}})| = |B_{n-1}^{D_2}(X^{\mathcal{B}})| = |B_2(X_1^{\mathcal{B}})|$$

since  $(2, 1, 1) \in \mathcal{F}$ . This shows that  $0 = h^{D_1}(\mathcal{B}) < h(\mathcal{B})$ . Therefore, we have  $(2, 1, 1) \in \mathcal{B}$ . Analogous argument demonstrates that  $0 = h^{D_1}(\mathcal{B}) < h(\mathcal{B})$  if  $\{(1, 2, 2), (2, 2, 2)\} \subseteq \mathcal{F}$ .

The proof is then complete.  $\square$

Theorem 1.6 demonstrates that the topological entropy of  $X$  is associated with a system of nonlinear recurrence equations

$$\begin{cases} a_n^{(1)} = F^{(1)}(a_{n-1}^{(1)}, a_{n-1}^{(2)}), \\ a_n^{(2)} = F^{(2)}(a_{n-1}^{(1)}, a_{n-1}^{(2)}), \\ a_2 = |B_2(X_1^{\mathcal{B}})|, b_2 = |B_2(X_2^{\mathcal{B}})|. \end{cases}$$

for some  $F^{(1)}$  and  $F^{(2)}$ , where  $a_n^{(i)} = |B_n(X_i^{\mathcal{B}})|$  for  $n \geq 3$  and  $i = 1, 2$ . Let  $v_i$  be the indicator vector of  $F^{(i)}$  for  $i = 1, 2$ . Theorem 4.3 indicates a sufficient condition for the coincidence of topological entropy of  $X$  with and without periodic boundary condition.

**Theorem 4.3.** *Suppose  $X^{\mathcal{B}}$  is a Markov tree-shift and  $h(\mathcal{B}) > 0$ .*

- (1)  *$h^P(\mathcal{B}) = h(\mathcal{B})$  if  $v_i$  dominates  $v_{\bar{i}}$  and  $\{(1, i, i), (2, i, i)\} \subseteq \mathcal{B}$  for some  $i = 1, 2$ , where  $i + \bar{i} = 3$ .*
- (2) *If  $h^P(\mathcal{B}) = h(\mathcal{B})$ , then  $\{(1, i, i), (2, i, i)\} \subseteq \mathcal{B}$  for some  $i = 1, 2$ .*

*Proof.* (a) Since  $v_i$  dominates  $v_{\bar{i}}$ , we have

$$h(\mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln^2 |B_n(X^{\mathcal{B}})|}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 |B_n(X_i^{\mathcal{B}})|}{n}.$$

Moreover,  $\{(1, i, i), (2, i, i)\} \subseteq \mathcal{B}$  infers that

$$|B_n^P(X^{\mathcal{B}})| \geq |B_n^P(X_i^{\mathcal{B}})| = |B_{n-1}(X_i^{\mathcal{B}})|.$$

The desired result then follows.

(b) Suppose, for each  $i \in \{1, 2\}$ , there exists  $i' \in \{1, 2\}$  such that  $(i', i, i) \in \mathcal{F}$ . A routine verification demonstrates that  $\{(i, 1, 1), (i, 2, 2)\} \subseteq \mathcal{B}$  and  $\{(\bar{i}, 1, 1), (\bar{i}, 2, 2)\} \subseteq \mathcal{F}$  yields  $h^P(\mathcal{B}) = 0 < h(\mathcal{B})$  since  $|B_n^P(X^{\mathcal{B}})| \leq 2$  for all  $n$ , which derives contradiction. Without loss of generality, we assume that

$$\{(1, 1, 1), (2, 2, 2)\} \subseteq \mathcal{B} \quad \text{and} \quad \{(2, 1, 1), (1, 2, 2)\} \subseteq \mathcal{F}.$$

We can conclude that  $|B_n^P(X^{\mathcal{B}})| = 2$  for all  $n$ , which leads to  $h^P(\mathcal{B}) = 0 < h(\mathcal{B})$ . Therefore,  $h^P(\mathcal{B}) = h(\mathcal{B})$  implies  $\{(1, i, i), (2, i, i)\} \subseteq \mathcal{B}$  for some  $i = 1, 2$ .  $\square$

## 5. CONCLUSION AND REMARKS

**5.1. Results.** This paper studies the complexity of tree-shifts of finite type via a well-known indicator like entropy. The main difference between the definition of entropy for classical symbolic dynamics and tree-shifts is the power that apply to the natural logarithmic function. The entropy of binary Markov tree-shifts over two symbols, i.e.,  $d = k = 2$ , is characterized completely. It turns out the entropy of binary Markov tree-shifts over two symbols is either 0 or  $\ln 2$ . In other words, the dynamical behavior of a binary Markov tree-shift with two symbols is either the simplest or the most complicated.

For the general case, i.e.,  $\max\{d, k\} \geq 3$ , we show that, the entropy set of tree-shifts of finite type covers a large number of reals which include the multinacci numbers (Realization Theorem). More importantly, such tree-shift can be constructed explicitly. Finally, the entropies of some tree-shifts of finite type can be computed explicitly (dominant-type, uniformly mixing and symmetric-type).

Whenever we restrict the case where  $d = k = 2$ , the necessary and sufficient condition for the coincidence of the entropy with and without boundary condition is elaborated. Herein, we consider three different types of boundary conditions, say, the periodic, Dirichlet, and Neumann boundary conditions. It is remarkable that the invariance of entropy with boundary constraints is determined by the basic set of allowed patterns.

**5.2. Open problems.** There are several interesting open problems related to this paper. Some of them are in preparation and will be discussed in the future work.

**Problem 2.** How to compute the hidden entropy explicitly for  $d = k = 2$ ?

**Problem 3.** How to compute the entropy for  $d, k \geq 2$ ?

To the best of our knowledge, there is no algorithm for solving the entropy and hidden entropy of an SNRE. Many researchers have devoted to the study of nonlinear recurrence equations. Readers are referred to [14] for more details.

**Problem 4.** Can the entropy of 1- $d$  (or 2- $d$ ) SFTs be realized by TSFT? That is, if  $\ln \rho$  is the topological entropy of some matrix shift  $X = X_A$  (or  $X = X_{A,B}$ ), where  $A$  and  $B$  are the corresponding transition matrices. Does there exist a TSFT with  $h(\mathcal{B}) = \ln \rho$ ? One can check that the number  $\rho$  in Theorem 3.1 is a class of topological entropy of 1- $d$  SFTs. But the construction of the TSFT in the proof of Theorem 3.1 can not extend to the general cases.

In [6], we show that every tree-shift of finite type is topological conjugate to a vertex tree-shift, which is defined analogously to the definition of matrix shifts. We conjecture that the answer to the above problem is affirmative, and the related work is in preparation.

Last but not least, Section 4 elaborates, for the case where  $d = k = 2$ , the necessary and sufficient conditions for the coincidence of entropy of Markov tree-shifts with and without boundary condition. Herein, three different types of frequently considered boundary conditions are discussed, that is to say, periodic, Dirichlet, and Neumann boundary conditions. Except for the invariance problem, it is natural to ask the following: How to compute the entropy of tree-shifts of finite type with boundary condition?

**Problem 5.** How to compute the exact value of the entropy of tree-shifts of finite type with boundary condition?

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